Parameterization for any line segment:

Starting at (a, b) and ending at (c, d):

$$r(t) = t\langle c, d\rangle + (1-t)\langle a, b\rangle$$

$$(0 \leq t \leq 1)$$

Example 1:

Find $\int_C f(x,y)\,dx$ and $\int_C f(x,y)\,dy$

where $f(x,y) = xe^y$ and

C is the line segment from $(2,3)$ to $(5,12)$

Parameterize C by

$$\vec{r}(t) = t\langle 5,12\rangle + (1-t)\langle 2,3\rangle$$

$$= \langle \underbrace{3t+2}_{x(t)}, \underbrace{9t+3}_{y(t)} \rangle$$

Then

$$\int_C f(x,y)\,dx$$

$= x'(t)$

$$= \int_0^1 (3t+2)\left(e^{9t+3}\right)\cdot 3\,dt$$

$$= 3e^3 \int_0^1 (3t+2)\,e^{9t}\,dt$$

Integrate by parts

$u = 3t+2$ $\qquad$ $v = e^{9t}/9$

$du = 3\,dt$ $\qquad$ $dv = e^{9t}\,dt$

Then

$$3e^3 \int_0^1 (3t+2)\, e^{9t}\, dt$$

$$= 3e^3 \left(\frac{(3t+2)e^{9t}}{9} \Big|_0^1 - \frac{3}{9} \int_0^1 e^{9t}\, dt \right)$$

$$= 3e^3 \left(\frac{(3t+2)e^{9t}}{9} - \frac{e^{9t}}{27} \right) \Big|_0^1$$

$$= 3e^3 \left(\left(\frac{5e^9}{9} - \frac{e^9}{27} \right) - \left(\frac{2}{9} - \frac{1}{27} \right) \right)$$

$$= 3e^3 \left(\frac{14e^9 - 5}{27} \right) = \boxed{\frac{e^3(14e^9 - 5)}{9}}$$

Similarly,

$$\int_C f(x,y)\,dy$$

$$= \int_0^1 (3t+2)e^{9t+3} \cdot \underbrace{9}_{=y'(t)}\,dt$$

$$= 9e^3 \underbrace{\int_0^1 (3t+2)e^{9t}\,dt}_{\text{same integral as before}}$$

$$= 9e^3 \left(\frac{14e^9-5}{27}\right) = \boxed{\frac{e^3(14e^9-5)}{3}}$$

3-D line integrals

Same formula, more variables

If C in 3-D is parameterized by

$(x(t), y(t), z(t)) = r(t)$

and is traced out no more than once from

$t=a$ to $t=b$

Example 2: Find

$$\int_C x^2 y z^3 \, ds$$

where C is parameterized by $\vec{r}(t) = \langle t^2, t, 1 \rangle$

from $t = 1$ to $t = 4$

$$\|r'(t)\| = \|\langle 2t, 1, 0 \rangle\|$$

$$= \sqrt{4t^2 + 1}$$

$$\int_C x^2 y z^3 \, ds$$

$$= \int_1^4 t^4 \cdot t \cdot 1 \sqrt{4t^2+1} \, dt$$

$$= \int_1^4 t^5 \sqrt{4t^2+1} \, dt$$

$$u = 4t^2 + 1 \longrightarrow t^2 = \frac{u-1}{4}$$

$$du = 8t \, dt$$

$$= \frac{1}{8} \int_5^{65} \left(\frac{u-1}{4}\right)^2 \sqrt{u} \, du$$

$$= \frac{1}{128} \int_{5}^{65} \left(u^{5/2} - 2u^{3/2} + u^{1/2} \right) u$$

$$= \frac{1}{128} \left(\frac{2u^{7/2}}{7} - \frac{4u^{5/2}}{5} + \frac{2u^{3/2}}{3} \right) \Bigg|_{5}^{65}$$

$$= \frac{u^{3/2}}{128} \left(\frac{2}{7} u^2 - \frac{4}{5} u + \frac{2}{3} \right) \Bigg|_{5}^{65}$$

$$= \boxed{\frac{(65)^{3/2}}{128} \left(\frac{24272}{21} \right) - \frac{5^{3/2}}{128} \left(\frac{80}{21} \right)}$$

Vector Fields (16.1)

To every point in a region R in $\mathbb{R}^n$, assign an n-dimensional vector. This is called a vector field. As usual, we will be concerned with $n=2$ or $n=3$.

How you make this precise

Later...

You can think of the vector field as a force field (gravitational, electromagnetic, nuclear, etc.) where the force "acts" on every point in the given region

You try to draw the vector field by plotting some points in the region, then drawing the vectors emanating from those points

Example 3: (electric field of point charge)

Assume there is a point charge at $(0,0,0)$ in $\mathbb{R}^3$. Then if $(x,y,z) \neq (0,0,0)$, the electric field is

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 (x^2+y^2+z^2)^{3/2}} \langle x,y,z \rangle$$

where $8.85 \times 10^{-12}\ C/Jm = \varepsilon_0$ is the permittivity of the region, q = charge strength

If we suppose the charge to be an electron,

$$q = -1.602 \times 10^{-19}\ C$$

If the charge is a proton,

$$q = 1.602 \times 10^{-19}\ C.$$

To make this easier to see, I'll assume a charge of 10,000 electrons

Mathematica

$$\text{VectorPlot3D}[\{\vec{F}\}, \{x, x_1, x_2\}, \{y, y_1, y_2\}, \{z, z_1, z_2\}]$$

Let $x_1 = y_1 = .01$, $z_1 = 1$, $x_2 = y_2 = 2$

Or choose your favorite numbers.

This command works in Wolfram Alpha.

Easy way to get a vector field: If $f: \mathbb{R}^n \to \mathbb{R}$,

assign to every point $(x_1, x_2, \ldots, x_n)$ the vector

$\nabla f(x_1, x_2, \ldots, x_n)$.

Example 4: Let $f(x,y) = \sin(\ln(y)x)$

for $x, y > 0$ The gradient

field is simply

$$\nabla f(x,y) = \left\langle \ln(y)\cos(\ln(y)x), \frac{x}{y}\cos(\ln(y)x) \right\rangle$$

Mathematica

VectorPlot[{∇f}, {x, x_1, x_2}, {y, y_1, y_2}]

Let $x_1 = y_1 = 1$, $x_2 = y_2 = 2$

or choose your favorite numbers - works in Wolfram Alpha

A vector field F on R in $\mathbb{R}^n$ is called conservative if there is a function $f: R \to \mathbb{R}$ with

$$F = \nabla f$$

for all points in R.

The function f is then called the potential of F

This should remind you of the fundamental theorem of calculus!

In fact, there is a fundamental theorem for line integrals against a vector field.

Fundamental Theorem for Line Integrals (vector field version)

Let $\vec{F}$ be a vector field on $\mathbb{R}^n$ and let C be a (piecewise) smooth curve in $\mathbb{R}^n$, parameterized by $\vec{r(t)}$, $a \leq t \leq b$.

Then

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

Furthermore, if $\vec{F}$ is conservative and $\vec{F} = \nabla f$, then

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))}$$

Note: If $\vec{F}$ is a force field, we can interpret the line integral as work along the path C against $\vec{F}$